

# How statistical are quantum states?

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A novel no-go theorem is presented which sets a bound upon the extent to which ‘ $\Psi$ -epistemic’ interpretations of quantum theory are able to explain the overlap between non-orthogonal quantum states in terms of an experimenter’s ignorance of an underlying state of reality. The theorem applies to any Hilbert space of dimension greater than two. In the limit of large Hilbert spaces, no more than half of the overlap between quantum states can be accounted for. Unlike other recent no-go theorems no additional assumptions, such as forms of locality, invasiveness, or non-contextuality, are required. The result continues to hold in the presence of a small but finite amount of noise, and is open to experimental verification in a sufficiently precise experimental arrangement.

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One of the most remarkable features of a quantum system is that of non-orthogonality: it is possible to prepare two different pure quantum states which cannot be perfectly discriminated by a single ideal measurement. No such property exists in classical systems, where two different classical states can always be perfectly discriminated by a suitably ideal measurement.

Statistical interpretations of the quantum state (see [1–4] and references therein) present a natural way to understand this feature of non-orthogonality. Suppose that a ‘quantum state’ is only a statistical state, associated with a probability distribution over some underlying physical states. Even an ideal quantum preparation procedure cannot uniquely fix this underlying state of the system but instead prepares one of a number of possible such states, with a well defined probability. In such an interpretation, when two quantum states are non-orthogonal, the associated probability distributions will overlap: the preparation procedures each have a non-zero probability of preparing the same physical states, and this is why they cannot be perfectly distinguished. This has been called a  $\Psi$ -epistemic interpretation [2, 3], as the idea is that the wavefunction does not itself represent a physical state, but instead represents an epistemic uncertainty about the physical state.

A number of recent no-go theorems [5–8] have explored the consequences of attempting to construct such  $\Psi$ -epistemic interpretations of quantum theory. This paper contributes a new no-go theorem, demonstrating that any attempt to explain non-orthogonality entirely in terms of an epistemic overlap is impossible: there are no ‘maximally epistemic’ interpretations of quantum theory. Using an empirically testable measure, it is shown that in the limit of large Hilbert spaces, no more than half of the overlap between quantum states could be explained in terms of an overlap of probability distributions. Un-

like the other recent results, the theorem presented here makes no additional assumptions beyond the definition of a  $\Psi$ -epistemic theory. This means that this theorem is the only such theorem to apply, for example, to the constructive  $\Psi$ -epistemic theory presented in [9].

*Ontological models and  $\Psi$ -epistemic theories.*— Two principal assumptions are involved in analysing whether a theory is  $\Psi$ -epistemic :

1. After a physical procedure prepares a quantum state,  $|\psi\rangle$ , the system is actually in a physical state  $\lambda$ . This physical state is not necessarily identified with the quantum state;
2. The probabilities of getting the results of a measurement procedure on the system are wholly determined by the physical state  $\lambda$  and the measurement procedure. The preparation procedure only influences the measurement outcomes indirectly, through the possible physical states prepared.

These assumptions are developed into the *ontological models* formalism in [3, 10–13], where the physical state  $\lambda$ , is referred to as an *ontic* state.

- A preparation procedure, which prepares the quantum state  $|\psi\rangle$ , will prepare some ontic state  $\lambda$  with probability  $\mu_\psi(\lambda)$ :

$$\begin{aligned} \mu_\psi(\lambda) &\geq 0 \\ \int \mu_\psi(\lambda) d\lambda &= 1 \end{aligned} \quad (1)$$

- A measurement,  $M$ , has a number of possible outcomes  $\{Q\}$ , and a probability  $\xi_M(Q|\lambda)$  of obtaining a particular outcome  $Q$ , given the ontic state  $\lambda$ :

$$\begin{aligned} \xi_M(Q|\lambda) &\geq 0 \\ \sum_Q \xi_M(Q|\lambda) &= 1 \end{aligned} \quad (2)$$

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- An ontic model will reproduce the results of quan-

tum theory<sup>1</sup> if, and only if:

$$\int \mu_\psi(\lambda) \xi_M(Q|\lambda) d\lambda = |\langle Q|\psi\rangle|^2 \quad (3)$$

Some immediate consequences of this formalism will be useful.

The set  $\Lambda_\phi = \{\lambda : \mu_\phi(\lambda) > 0\}$  is the set of all possible ontic states which may occur when preparing the quantum state  $|\phi\rangle$ .

If one of the outcomes of the measurement  $M$  is represented by the projection onto  $|\phi\rangle$ , then  $\int \mu_\phi(\lambda) \xi_M(\phi|\lambda) d\lambda = |\langle \phi|\phi\rangle|^2 = 1$ . It follows that

$$\forall \lambda \in \Lambda_\phi \quad \xi_M(\phi|\lambda) = 1 \quad (4)$$

If  $|\varphi\rangle$  is orthogonal to  $|\phi\rangle$  then  $\int \mu_\varphi(\lambda) \xi_M(\phi|\lambda) d\lambda = |\langle \phi|\varphi\rangle|^2 = 0$ . It follows that

$$\forall \lambda \in \Lambda_\varphi \quad \xi_M(\phi|\lambda) = 0 \quad (5)$$

which implies<sup>2</sup>  $\Lambda_\varphi \cap \Lambda_\phi = \emptyset$ .

If the quantum two states  $|\psi\rangle$  and  $|\phi\rangle$  are non-orthogonal, then:

$$\int \mu_\psi(\lambda) \xi_M(\phi|\lambda) d\lambda = |\langle \phi|\psi\rangle|^2 \neq 0 \quad (6)$$

This allows the possibility that  $\Lambda_\psi \cap \Lambda_\phi$  is not empty.

A  $\Psi$ -epistemic theory is defined[3] to be one in which there exist at least two distinct non-orthogonal quantum states,  $|\psi\rangle$  and  $|\phi\rangle$ , for which  $\Lambda_\psi \cap \Lambda_\phi \neq \emptyset$ . In such a theory, the fact that the quantum states  $|\psi\rangle$  and  $|\phi\rangle$  cannot be discriminated by a single ideal measurement is, at least partially, explained by the fact that there is some  $\lambda \in \Lambda_\psi \cap \Lambda_\phi$ . As all the results of any measurement  $M$  are determined by  $\xi_M(Q|\lambda)$ , and both  $|\psi\rangle$  and  $|\phi\rangle$  can prepare ontic states in  $\Lambda_\psi \cap \Lambda_\phi$ , it follows that there is no possible measurement outcome which can occur with zero probability for  $|\psi\rangle$  but with certainty for  $|\phi\rangle$  (and vice versa).

The term  $\int_{\Lambda_\phi} \mu_\psi(\lambda) d\lambda$  gives a measure of the epistemic overlap between the quantum states  $|\psi\rangle$  and  $|\phi\rangle$ . This is bound by the measure of the quantum state overlap. As  $\forall \lambda \in \Lambda_\phi, \xi_M(\phi|\lambda) = 1$

$$\begin{aligned} \int_{\Lambda_\phi} \mu_\psi(\lambda) d\lambda &= \int_{\Lambda_\phi} \mu_\psi(\lambda) \xi_M(\phi|\lambda) d\lambda \\ &\leq \int \mu_\psi(\lambda) \xi_M(\phi|\lambda) d\lambda = |\langle \phi|\psi\rangle|^2 \end{aligned} \quad (7)$$

The degree of epistemic overlap,  $\Omega[\phi, \psi]$ , between two states can now be defined<sup>3</sup> via:

$$\int_{\Lambda_\phi} \mu_\psi(\lambda) d\lambda = \Omega[\phi, \psi] |\langle \phi|\psi\rangle|^2 \quad (8)$$

A maximally  $\Psi$ -epistemic theory would have  $\Omega[\phi, \psi] = 1$  for all pairs of quantum states. The quantum state overlap  $|\langle \phi|\psi\rangle|^2$  would be accounted for entirely in terms of the overlap in the ontic states  $\Lambda_\psi \cap \Lambda_\phi$ .

By contrast, a maximally  $\Psi$ -ontic theory is one in which  $\Lambda_\psi \cap \Lambda_\phi = \emptyset$  for all pairs of distinct quantum states, and so  $\Omega[\phi, \psi] = 0$ . In such a theory, no two distinct quantum state preparations  $|\psi\rangle$  and  $|\phi\rangle$  could possibly produce the same ontic state  $\lambda$ . For a maximally  $\Psi$ -ontic theory, therefore, given the ontic state  $\lambda$  it is always possible to identify the quantum state  $|\psi\rangle$ . The de Broglie-Bohm[14, 15], Everett[16] and CSL[17] interpretations of quantum theory are all examples of maximally  $\Psi$ -ontic theories.

It will now be shown that, in Hilbert spaces of dimension greater than two, there are no models of quantum theory for which  $\Omega[\phi, \psi] = 1$  for all states: there are no maximally  $\Psi$ -epistemic interpretations. Making a symmetrical assumption that  $\Omega[\phi, \psi]$  is a constant,  $\Omega$ , between all pairs of distinct states it will be shown that  $\Omega \leq \frac{9}{10}$  for a 3 dimensional Hilbert space, falling to  $\Omega \leq \frac{1}{2}$  in the limit of large Hilbert spaces. No more than 50% of the quantum state overlap can be accounted for by an epistemic overlap of probability distributions.

*Three dimensional Hilbert space.*- Consider the following states<sup>4</sup> of a three dimensional Hilbert space:

$$\begin{aligned} |p\rangle &= \frac{1}{\sqrt{3}}(|a\rangle + |b\rangle + |c\rangle), \quad |m\rangle = \frac{1}{\sqrt{3}}(|a\rangle + |b\rangle - |c\rangle) \\ |a_+\rangle &= \frac{1}{\sqrt{2}}(|a\rangle + |c\rangle), \quad |a_-\rangle = \frac{1}{\sqrt{2}}(|a\rangle - |c\rangle) \\ |b_+\rangle &= \frac{1}{\sqrt{2}}(|b\rangle + |c\rangle), \quad |b_-\rangle = \frac{1}{\sqrt{2}}(|b\rangle - |c\rangle) \end{aligned}$$

where  $|a\rangle, |b\rangle, |c\rangle$  form an orthonormal basis, with the three measurements:

$$\begin{aligned} M_1 &: |a_+\rangle \langle a_+|, |a_-\rangle \langle a_-|, |b\rangle \langle b| \\ M_2 &: |b_+\rangle \langle b_+|, |b_-\rangle \langle b_-|, |a\rangle \langle a| \\ M_3 &: |a\rangle \langle a|, |b\rangle \langle b|, |c\rangle \langle c| \end{aligned}$$

The results are shown in Table I. From  $M_1$  it can be seen that

$$\begin{aligned} \forall \lambda \in \Lambda_a, \quad \xi_{M_1}(b|\lambda) &= 0 \\ \forall \lambda \in \Lambda_p, \quad \xi_{M_1}(a_-|\lambda) &= 0 \\ \forall \lambda \in \Lambda_m, \quad \xi_{M_1}(a_+|\lambda) &= 0 \end{aligned} \quad (9)$$

For any given  $\lambda$ , it must be that  $\xi_{M_1}(b|\lambda) + \xi_{M_1}(a_-|\lambda) + \xi_{M_1}(a_+|\lambda) = 1$ , so this means:

$$\Lambda_a \cap \Lambda_p \cap \Lambda_m = \emptyset \quad (10)$$

Similarly

$$\Lambda_c \cap \Lambda_p \cap \Lambda_m = \emptyset \quad (11)$$

<sup>1</sup> Additional structure is required to represent unitary operations, but these are not needed for this proof.

<sup>2</sup> Ignoring sets of measure zero.

<sup>3</sup>  $\Omega[\psi, \psi] = 1$  by definition. Where  $\langle \phi|\psi\rangle = 0$ ,  $\Omega[\phi, \psi]$  may take any value.

<sup>4</sup> These are also the states used in Clifton's proof of quantum contextuality[18]. For further connections to quantum contextuality, see[19].

(a)

	$M_1$		
	$ b\rangle\langle b $	$ a_+\rangle\langle a_+ $	$ a_-\rangle\langle a_- $
$ a\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$
$ b\rangle$	1	0	0
$ c\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$
$ p\rangle$	$\frac{1}{3}$	$\frac{2}{3}$	0
$ m\rangle$	$\frac{1}{3}$	0	$\frac{2}{3}$

(b)

	$M_2$		
	$ a\rangle\langle a $	$ b_+\rangle\langle b_+ $	$ b_-\rangle\langle b_- $
$ a\rangle$	1	0	0
$ b\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$
$ c\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$
$ p\rangle$	$\frac{1}{3}$	$\frac{2}{3}$	0
$ m\rangle$	$\frac{1}{3}$	0	$\frac{2}{3}$

(c)

	$M_3$		
	$ a\rangle\langle a $	$ b\rangle\langle b $	$ c\rangle\langle c $
$ a\rangle$	1	0	0
$ b\rangle$	0	1	0
$ c\rangle$	0	0	1
$ p\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$ m\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

TABLE I: Measurement outcomes

$M_2$  adds

$$\Lambda_b \cap \Lambda_p \cap \Lambda_m = \emptyset \quad (12)$$

leading to

$$(\Lambda_a \cup \Lambda_b \cup \Lambda_c) \cap \Lambda_p \cap \Lambda_m = \emptyset \quad (13)$$

If there is any overlap in  $\Lambda_p \cap \Lambda_m$  then it must lie outside  $\Lambda_a \cup \Lambda_b \cup \Lambda_c$ .

Now  $M_3$  gives

$$\int_{\Lambda_a} \mu_p(\lambda) d\lambda = \frac{\Omega[a, p]}{3} \quad (14)$$

and similarly for  $|b\rangle$  and  $|c\rangle$ . As  $\Lambda_b \cap \Lambda_a = \emptyset$  etc.

$$\int_{(\Lambda_a \cup \Lambda_b \cup \Lambda_c)} \mu_p(\lambda) d\lambda = \frac{(\Omega[a, p] + \Omega[b, p] + \Omega[c, p])}{3} \quad (15)$$

It follows that:

$$\int_{\Lambda_m} \mu_p(\lambda) d\lambda \leq 1 - \frac{(\Omega[a, p] + \Omega[b, p] + \Omega[c, p])}{3} \quad (16)$$

but

$$\int_{\Lambda_m} \mu_p(\lambda) d\lambda = \frac{\Omega[m, p]}{9} \quad (17)$$

giving

$$\frac{\Omega[m, p]}{9} \leq 1 - \frac{(\Omega[a, p] + \Omega[b, p] + \Omega[c, p])}{3} \quad (18)$$

If the ontic model has maximal epistemic overlaps with the  $|a\rangle, |b\rangle, |c\rangle$  states, then  $\Omega[a, p] = \Omega[b, p] = \Omega[c, p] = 1$ , but this requires  $\Omega[m, p] = 0$  while  $|\langle m | p \rangle|^2 = \frac{1}{9}$ . This proves that a maximally  $\Psi$ -epistemic theory is not possible.

Reducing the degree of epistemicness in the  $|a\rangle, |b\rangle, |c\rangle$  basis will allow some epistemic overlap between  $|p\rangle$  and  $|m\rangle$ . For example, keeping  $\Omega[a, p] = \Omega[b, p] = 1$  it is possible to reach  $\Omega[m, p] = 1$  with  $\Omega[c, p] = \frac{2}{3}$ . More symmetrically,  $\Omega[m, p] = 1$  is possible with  $\Omega[a, p] = \Omega[b, p] = \Omega[c, p] = \frac{8}{9}$ .

A basis independent measure of how epistemic a theory can be and still reproduce quantum statistics is given by letting  $\Omega[\phi, \psi] = \Omega$  be a constant for all non-orthogonal states. This gives:

$$\Omega \leq \frac{9}{10} \quad (19)$$

*Higher dimensional Hilbert spaces.*- The above results can be generalised to Hilbert spaces of dimension  $d > 3$ . In the limit of large  $d$ ,  $\Omega \leq \frac{1}{2}$ .

Take the basis states

$$|a_1\rangle, |a_2\rangle, \dots, |a_d\rangle$$

the superpositions

$$\begin{aligned} |a_{i+}\rangle &= \frac{1}{\sqrt{2}}(|a_i\rangle + |a_d\rangle) \\ |a_{i-}\rangle &= \frac{1}{\sqrt{2}}(|a_i\rangle - |a_d\rangle) \\ |p_d\rangle &= \frac{1}{\sqrt{d}} \sum_{i=1, d} |a_i\rangle \\ |m_d\rangle &= \frac{1}{\sqrt{d}} \left( \sum_{i=1, d-1} |a_i\rangle \right) - \frac{1}{\sqrt{d}} |a_d\rangle \end{aligned}$$

and the measurements

$$\begin{aligned} M_i : & |a_1\rangle\langle a_1|, \dots, |a_{i-1}\rangle\langle a_{i-1}|, \\ & |a_{i+1}\rangle\langle a_{i+1}|, \dots, |a_{d-1}\rangle\langle a_{d-1}|, \\ & |a_{i+}\rangle\langle a_{i+}|, |a_{i-}\rangle\langle a_{i-}| \\ M_d : & |a_1\rangle\langle a_1|, \dots, |a_d\rangle\langle a_d| \end{aligned}$$

Each measurement  $M_i$  now shows:

$$\begin{aligned} \Lambda_{a_i} \cap \Lambda_{p_d} \cap \Lambda_{m_d} &= \emptyset \\ \Lambda_{a_d} \cap \Lambda_{p_d} \cap \Lambda_{m_d} &= \emptyset \end{aligned} \quad (20)$$

leading to

$$(\cup_i \Lambda_{a_i}) \cap \Lambda_{p_d} \cap \Lambda_{m_d} = \emptyset \quad (21)$$

From the  $M_d$  measurement:

$$\int_{(\cup_i \Lambda_{a_i})} \mu_{p_d}(\lambda) d\lambda = \sum_i \Omega[a_i, p_d] |\langle a_i | p_d \rangle|^2 = \frac{1}{d} \sum_i \Omega[a_i, p_d] \quad (22)$$

so

$$\int_{\Lambda_{m_d}} \mu_{p_d}(\lambda) d\lambda \leq 1 - \frac{1}{d} \sum_i \Omega[a_i, p_d] \quad (23)$$

giving

$$\Omega[m_d, p_d] \left(1 - \frac{2}{d}\right)^2 \leq 1 - \frac{1}{d} \sum_i \Omega[a_i, p_d] \quad (24)$$

Once again,  $\Omega[a_i, p_d] = 1$  would require  $\Omega[m_d, p_d] = 0$ , even though  $|\langle m_d | p_d \rangle|^2 = \left(1 - \frac{2}{d}\right)^2 \rightarrow 1$ . For a theory to be maximally  $\Psi$ -epistemic in a specific basis, then as  $d \rightarrow \infty$  it must become maximally  $\Psi$ -ontic between at least some states arbitrarily close to each other. To obtain a full overlap  $\Omega[m_d, p_d] = 1$  by symmetrically reducing the overlap with the  $|a_i\rangle$  basis would require  $\Omega[a_i, p_d] = \frac{4}{d^2}(d-1) \rightarrow 0$ .

The basis independent measure of how epistemic this allows the theory overall sets  $\Omega[\phi, \psi] = \Omega(d)$  as before giving

$$\Omega(d) \leq \frac{d^2}{2d^2 - 4d + 4} \quad (25)$$

In the limit  $d$  gets large,  $\Omega(d) \leq \frac{1}{2}$ .

*Experimental Errors.*- The ideal measurement depends upon obtaining zero probability of a set of outcomes. In any real experiment, there will be some noise, giving a value  $\epsilon > 0$ . We follow [5] in introducing a measure of overlap in the ontic state distribution which can be compared to experimental data in the presence of noise:

$$\int \min[\mu_\phi(\lambda), \mu_\psi(\lambda)] d\lambda = \omega[\phi, \psi] |\langle \phi | \psi \rangle|^2 \quad (26)$$

$\omega[\phi, \psi]$  replaces  $\Omega[\phi, \psi]$  as the measure of the degree to which the quantum state overlap is explained by the overlap in probability distributions. It is shown in the appendix that when the errors are of order  $\epsilon$ , the bound for a 3 dimensional Hilbert space becomes

$$\frac{\omega[m, p]}{9} \leq 1 - \frac{(\omega[a, p] + \omega[b, p] + \omega[c, p])}{3} + 18\epsilon \quad (27)$$

For a constant  $\omega$ ,

$$\omega \leq \frac{9}{10} + \frac{162}{10}\epsilon \quad (28)$$

requiring  $\epsilon < \frac{1}{162}$  to rule out maximally  $\Psi$ -epistemic theories.

For Hilbert spaces of dimension  $d$ ,

$$\omega[p_d, m_d] \left(1 - \frac{2}{d}\right)^2 \leq 1 - \frac{1}{d} \sum_i \omega[p_d, a_i] + \frac{1}{2}d^2(d+1)\epsilon \quad (29)$$

For a constant  $\omega(d)$ ,

$$\omega(d) \leq \frac{d^2 + \frac{1}{2}d^4(d+1)\epsilon}{2d^2 - 4d + 4} \quad (30)$$

requiring

$$\epsilon < \frac{2(d-2)^2}{d^4(d+1)} \quad (31)$$

In the limit of large  $d$ , this gives

$$\omega(d) \leq \frac{1}{2} + \epsilon \frac{d^3}{4} \quad (32)$$

requiring  $\epsilon < \frac{2}{d^3}$ .

Although this requires a very high precision experimental test, sufficiently small finite  $\epsilon$  will nevertheless ensure that  $\omega(d) \leq 1$  and in large Hilbert spaces  $\omega(d) \leq \frac{1}{2}$ . The theorem is therefore robust against finite precision loopholes.

*Conclusions.*- There exist constructive examples of  $\Psi$ -epistemic models, which can reproduce all the statistics of quantum theory (such as the LJBR model[9]). It is therefore not the intention of this article to attempt to construct no-go theorems which aim to show that  $\Psi$ -epistemic theories are impossible simpliciter. Existing no-go theorems have effectively sought to prove  $\Omega[\phi, \psi] = 0$ , but to do so must assume additional properties, such as forms of locality[8], factorisability of product states[5] or non-invasiveness[7]. This should not downplay the significance of these theorems: any attempt to construct  $\Psi$ -epistemic theories must necessarily reject these properties, in much the same way that Bell's theorem[20] requires realist interpretations of quantum theory to reject local causality. They help to map out the space of possible theories which can account for quantum phenomena.

The no-go theorem presented here contributes to this task. It does not make additional assumptions so applies even to theories that reject all of these additional assumptions, such as the LJBR model. The LJBR model demonstrates that  $\Omega[\phi, \psi] > 0$  is indeed possible. This no-go theorem shows that there are empirically verifiable bounds upon how large  $\Omega[\phi, \psi]$  can become, even if all other assumptions are dropped. In the limit of large Hilbert spaces, no more than half the quantum state overlap can be accounted for in terms of epistemic overlaps. It is not clear that the LJBR model reaches the boundary derived in this paper. This would leave open two possibilities: either a more epistemic model than LJBR can be constructed, or else a tighter bound may be derived.

The question of whether the quantum state is real or statistical has been a matter of debate since the early days of quantum theory. It would be amusing if the answer turned out to be: 'about half and half'.

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## APPENDIX

**Error free proof.**— The error free proof involves decomposing the overlap between the  $|p\rangle$  and  $|m\rangle$  states:

$$\begin{aligned} P \cap M &= \\ & (P \cap M \cap (A \cup B \cup C)) \cup (P \cap M \cap \neg(A \cup B \cup C)) \\ &= (P \cap M \cap (A \cup B \cup C)) \\ & \quad \cup ((P \cap \neg(A \cup B \cup C)) \cap (M \cap \neg(A \cup B \cup C))) \end{aligned} \quad (33)$$

In the ideal case, for any measure  $\nu[X]$ , we have  $\nu[P \cap M \cap (A \cup B \cup C)] = 0$  and

$$\nu[P \cap M] \leq \min(\nu[P] - \nu[P \cap (A \cup B \cup C)], \quad (34)$$

$$\nu[M] - \nu[M \cap (A \cup B \cup C)]) \quad (35)$$

As

$$\begin{aligned} \nu[P \cap (A \cup B \cup C)] &= \nu[P \cap A] + \nu[P \cap B] + \nu[P \cap C] \\ & \quad - \nu[P \cap A \cap B] - \nu[P \cap B \cap C] - \nu[P \cap A \cap C] \\ & \quad + \nu[P \cap A \cap B \cap C] \end{aligned} \quad (36)$$

and  $\nu[P \cap A \cap B] = 0$  (and similar combinations) then

$$\begin{aligned} \nu[P \cap M] &\leq \\ & \min(\nu[P] - \nu[P \cap A] - \nu[P \cap B] - \nu[P \cap C], \\ & \quad \nu[M] - \nu[M \cap A] - \nu[M \cap B] - \nu[M \cap C]) \end{aligned} \quad (37)$$

Using the measure  $\mu[P \cap M] = \Omega[p, m] |\langle p | m \rangle|^2$  etc., we recover the ideal result.

**Noise tolerance.**— This proof does not appear noise tolerant, as it requires some measurement outcomes to have a probability of zero. We now show how to generalise the result to situations where the error in such measurements is  $\epsilon > 0$ .

In general, for any measure  $\nu[X]$ :

$$\begin{aligned} \nu[P \cap M \cap (A \cup B \cup C)] &= \\ & \nu[P \cap M \cap A] + \nu[P \cap M \cap B] + \nu[P \cap M \cap C] \\ & \quad - \nu[P \cap M \cap A \cap B] - \nu[P \cap M \cap B \cap C] \\ & \quad - \nu[P \cap M \cap A \cap C] + \nu[P \cap M \cap A \cap B \cap C] \end{aligned} \quad (38)$$

and

$$\begin{aligned} \nu[P \cap M \cap A \cap B] + \nu[P \cap M \cap B \cap C] \\ + \nu[P \cap M \cap A \cap C] \geq \nu[P \cap M \cap A \cap B \cap C] \end{aligned} \quad (39)$$

The worst case scenario is

$$\begin{aligned} \nu[P \cap M \cap (A \cup B \cup C)] &\leq \nu[P \cap M \cap A] \\ & \quad + \nu[P \cap M \cap B] + \nu[P \cap M \cap C] \end{aligned} \quad (40)$$

For

$$\begin{aligned} \nu[P \cap \neg(A \cup B \cup C)] &= \nu[P] - \nu[P \cap A] - \nu[P \cap B] \\ & \quad - \nu[P \cap C] + \nu[P \cap A \cap B] + \nu[P \cap B \cap C] \\ & \quad + \nu[P \cap A \cap C] - \nu[P \cap A \cap B \cap C] \end{aligned} \quad (41)$$

the worst case scenario is

$$\begin{aligned} \nu[P \cap \neg(A \cup B \cup C)] &= \nu[P] - \nu[P \cap A] - \nu[P \cap B] \\ & \quad - \nu[P \cap C] + \nu[P \cap A \cap B] + \nu[P \cap B \cap C] \\ & \quad + \nu[P \cap A \cap C] \end{aligned} \quad (42)$$

giving an expression suitable for non-zero overlaps:

$$\begin{aligned} \nu[P \cap M] &\leq \nu[P \cap M \cap A] + \nu[P \cap M \cap B] + \nu[P \cap M \cap C] \\ & \quad + \min[(\nu[P] - \nu[P \cap A] - \nu[P \cap B] - \nu[P \cap C]) \\ & \quad + \nu[P \cap A \cap B] + \nu[P \cap B \cap C] + \nu[P \cap A \cap C]), \\ & \quad (\nu[M] - \nu[M \cap A] - \nu[M \cap B] - \nu[M \cap C]) \\ & \quad + \nu[M \cap A \cap B] + \nu[M \cap B \cap C] + \nu[M \cap A \cap C])] \end{aligned} \quad (43)$$

**Measure of epistemic overlap.**— We now need a measure which can be compared to experimental data. We follow [5] in using:

$$\nu[\Phi \cap \Psi] = \int \min[\mu_\phi(\lambda), \mu_\psi(\lambda)] d\lambda \quad (44)$$

It can be straightforwardly verified that this generalises using:

$$\begin{aligned} \nu[\Phi_i] &= \int \mu_{\phi_i}(\lambda) d\lambda = 1 \\ \nu[\cap_i \Phi_i] &= \int \min_i [\mu_{\phi_i}(\lambda)] d\lambda \\ \nu[\cup_i \Phi_i] &= \int \max_i [\mu_{\phi_i}(\lambda)] d\lambda \\ \nu[\Phi \cap \neg\Psi] &= \int (\mu_\phi(\lambda) - \min[\mu_\phi(\lambda), \mu_\psi(\lambda)]) d\lambda \\ &= \nu[\Phi] - \nu[\Phi \cap \Psi] \end{aligned} \quad (45)$$

to include all the required measure theoretic relations such as  $\nu[A \cup B] = \nu[A] + \nu[B] - \nu[A \cap B]$  and  $\nu[A \cap (B \cup C)] = \nu[(A \cap B) \cup (A \cap C)]$  etc.

**Degree of epistemic overlap.**— For any  $\xi_M(Q|\lambda)$

$$\int \min[\mu_\phi(\lambda), \mu_\psi(\lambda)] \xi_M(Q|\lambda) d\lambda \leq \int \mu_\phi(\lambda) \xi_M(Q|\lambda) d\lambda \quad (46)$$

and

$$\int \min[\mu_\phi(\lambda), \mu_\psi(\lambda)] \xi_M(Q|\lambda) d\lambda \leq \int \mu_\psi(\lambda) \xi_M(Q|\lambda) d\lambda \quad (47)$$

so

$$\begin{aligned} \int \min[\mu_\phi(\lambda), \mu_\psi(\lambda)] \xi_M(Q|\lambda) d\lambda \\ \leq \min\left[\int \mu_\psi(\lambda) \xi_M(Q|\lambda) d\lambda, \int \mu_\phi(\lambda) \xi_M(Q|\lambda) d\lambda\right] \end{aligned} \quad (48)$$

As  $\sum_Q \xi_M(Q|\lambda) = 1$ ,

$$\begin{aligned} \nu[\Phi \cap \Psi] &= \sum_Q \int \min[\mu_\phi(\lambda), \mu_\psi(\lambda)] \xi_M(Q|\lambda) d\lambda \\ &\leq \sum_Q \min\left[\int \mu_\phi(\lambda) \xi_M(Q|\lambda) d\lambda, \int \mu_\psi(\lambda) \xi_M(Q|\lambda) d\lambda\right] \end{aligned} \quad (49)$$

This will be true for any measurement procedure  $M$ .

If the  $M$  measurement is a projection onto  $|\psi\rangle$  and its orthogonal subspace,

$$\sum_Q \min\left[\int \mu_\phi(\lambda) \xi_M(Q|\lambda) d\lambda, \int \mu_\psi(\lambda) \xi_M(Q|\lambda) d\lambda\right] = |\langle \phi | \psi \rangle|^2 \quad (50)$$

hence  $\nu[\Phi \cap \Psi] \leq |\langle \phi | \psi \rangle|^2$ . It can easily be verified that  $\nu[\Phi \cap \Psi] = 0$  if and only if  $\Lambda_\phi \cap \Lambda_\psi = \emptyset$ , and  $\nu[\Phi \cap \Psi] = 1$  if and only if  $\mu_\phi(\lambda) = \mu_\psi(\lambda)$ .

We can now define a degree of epistemic overlap  $\omega[\phi, \psi]$  that can be empirically bounded even in the presence of noise:

$$\nu[\Phi \cap \Psi] = \omega[\phi, \psi] |\langle \phi | \psi \rangle|^2 \quad (51)$$

**Comparison to experimental data.**— To compare to experimental data, if  $f_M(Q|\phi)$  is the measured frequency with which an outcome  $Q$  occurs, given the quantum state is  $|\phi\rangle$ , then

$$\nu[\Phi \cap \Psi] \leq \sum_Q \min[f_M(Q|\phi), f_M(Q|\psi)] \quad (52)$$

For the intersections between more than one quantum state, this generalises to

$$\begin{aligned} \nu[\cap_i \Phi_i] &\leq \sum_Q \min_i \left[ \int \mu_{\phi_i}(\lambda) \xi_M(Q|\lambda) d\lambda \right] \\ &\leq \sum_Q \min_i [f_M(Q|\phi_i)] \end{aligned} \quad (53)$$

where the terms on the right hand side can be read off directly from the measurement results.

Again, we find  $\nu[\cap_i \Phi_i] = 0$  if and only if  $\cap_i \Lambda_{\phi_i} = \emptyset$ . However, when there is noise in the measurement, each  $Q$  outcome will contribute  $\epsilon$  in place of a zero, so in a  $d$ -dimensional Hilbert space,  $\nu[\cap_i \Phi_i] \leq d\epsilon$ .

**Noise tolerant proof.**— For a three dimensional Hilbert space, this gives:

$$\begin{aligned} \omega[m, p] |\langle m | p \rangle|^2 &\leq 1 - \omega[a, p] |\langle a | p \rangle|^2 - \omega[b, p] |\langle b | p \rangle|^2 \\ &\quad - \omega[c, p] |\langle c | p \rangle|^2 + 18\epsilon \end{aligned} \quad (54)$$

and so

$$\frac{\omega[m, p]}{9} \leq 1 - \frac{\omega[a, p] + \omega[b, p] + \omega[c, p]}{3} + 18\epsilon \quad (55)$$

For a constant  $\omega$ ,

$$\omega \leq \frac{9}{10} + \frac{162}{10}\epsilon \quad (56)$$

Generalising to  $d$  dimensional Hilbert spaces is straightforward.

$$P \cap M = (P \cap M \cap (\cup_i A_i)) \cup (P \cap M \cap \neg(\cup_i A_i)) \quad (57)$$

The worst case scenario is

$$\nu[P \cap M \cap (\cup_i A_i)] \leq \sum_i \nu[P \cap M \cap A_i] \quad (58)$$

and

$$\begin{aligned} \nu[P \cap \neg(\cup_i A_i)] &\leq \nu[P] - \sum_i \nu[P \cap A_i] \\ &\quad + \frac{1}{2} \sum_{i \neq j} \nu[P \cap A_i \cap A_j] \end{aligned} \quad (59)$$

giving

$$\begin{aligned} \nu[P \cap M] &\leq \sum_i \nu[P \cap M \cap A_i] \\ &\quad + \min \left[ \left( \nu[P] - \sum_i \nu[P \cap A_i] + \frac{1}{2} \sum_{i \neq j} \nu[P \cap A_i \cap A_j] \right), \right. \\ &\quad \left. \left( \nu[M] - \sum_i \nu[M \cap A_i] + \frac{1}{2} \sum_{i \neq j} \nu[M \cap A_i \cap A_j] \right) \right] \end{aligned} \quad (60)$$

Each of the  $d$  terms  $\nu[P \cap M \cap A_i]$  and the  $\frac{1}{2}d(d-1)$  terms  $\nu[P \cap A_i \cap A_j]$  (or  $\nu[M \cap A_i \cap A_j]$ ) contribute terms of order  $d\epsilon$ , giving

$$\omega[p_d, m_d] |\langle p_d | m_d \rangle|^2 \leq 1 - \sum_i \omega[p_d, a_i] |\langle p_d | a_i \rangle|^2 + \frac{1}{2}d^2(d+1)\epsilon \quad (61)$$

so

$$\omega[p_d, m_d] \left(1 - \frac{2}{d}\right)^2 \leq 1 - \frac{1}{d} \sum_i \omega[p_d, a_i] + \frac{1}{2}d^2(d+1)\epsilon \quad (62)$$

For a constant  $\omega(d)$ ,

$$\omega(d) \leq \frac{d^2 + \frac{1}{2}d^4(d+1)\epsilon}{2d^2 - 4d + 4} \quad (63)$$

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